TEMPERATURE FIELD OF A TWO-LAYERED PLATE WITH TIME-VARYING HEAT-TRANSFER CONDITIONS

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The investigation results have shown that the solution to the heat conduction problem in a double-layer plate with time-variable boundary conditions may be obtained numerically if the solutions to the problem at constant boundary conditions are known. It is also shown that the method [8] is applicable for solving the problem on a nonstationary temperature field of the system of bodies at time-variable boundary conditions.

A solution of the problem of a two-layered plate with the layer of lower thermal conductivity facing the heating medium and the free side of the other layer assumed to be adiabatic is given in [1, 2]. In [3] the solution of [2] is extended to the case in which the heat input is on the side of greater thermal conductivity and the free side of the second layer is adiabatic. The roots of the characteristic equation are found, and the problem is solved completely. The solution and characteristic equation given in [4] are identical with those in [3] except that only the special case is treated in which the layer thicknesses are so small that the solution can be limited to the first few terms of the series.

The results in [4] stating that in the case of a thin layer of thermal insulation material the temperature distribution in the metal layer is a function of the time only enabled the authors of [5, 6] to solve the problem of the temperature distribution along the insulation thickness.

The solution of the problem of the temperature field in a two-layered plate with heat transfer at both boundaries is discussed in [7]. The solution and characteristic equation of [7] revert to those of [3] under the assumption of zero heat transfer at one of the boundaries ($\mathrm{Bi}_{\mathrm{II}} = 0$).

In all of the papers cited thus far, however, special cases of the problem of the nonstationary temperature field in a two-layered plate have been investigated.

In its most general context the problem for a two-layered plate with layer thicknesses δ_1 and δ_2 may be stated as follows:

$$\frac{\partial t_1(x,\ \tau)}{\partial \tau} = a_1 \frac{\partial^2 t_1(x,\ \tau)}{\partial x^2}, \quad -\delta_1 \leqslant x \leqslant 0, \tag{1a}$$

$$\frac{\partial t_2(x,\ \tau)}{\partial \tau} = a_2 \frac{\partial^2 t_2(x,\ \tau)}{\partial x^2}, \quad 0 \leqslant x \leqslant \delta_2 \tag{1b}$$

under the boundary conditions

$$-\left.\frac{\partial t_1(x,\ \tau)}{\partial x}\right|_{x=-\delta_1} = h_1(\tau)\left[t_{c_1}(\tau) - t_1(x,\ \tau)\right]\Big|_{x=-\delta_1}, \tag{2a}$$

$$-\left.\frac{\partial t_2(x,\ \tau)}{\partial x}\right|_{x=\delta_2} = h_2(\tau)\left[t_2(x,\ \tau) - t_{c_2}(\tau)\right]\Big|_{x=\delta_2}, \tag{2b}$$

$$\lambda_1 \left.\frac{\partial t_1(x,\ \tau)}{\partial x}\right|_{x=0} = \lambda_2 \left.\frac{\partial t_2(x,\ \tau)}{\partial x}\right|_{x=0}, \tag{2c}$$

$$t_1(x,\ \tau)_{x=0} = t_2(x,\ \tau)_{x=0}, \tag{2d}$$

$$t_1(x,\ 0) = t_2(x,\ 0) = t_0. \tag{2e}$$

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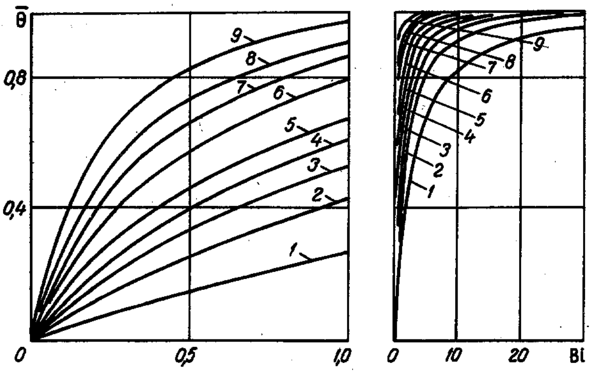

Fig. 1. Functions $\bar{\Theta} = f(\mathrm{Bi})$ determined from the parameter Fo_2 for a point on the surface of the higher-thermal-conductivity layer, $x/\delta_1 = 1$; $\rho = 0.1$: 1) $\mathrm{Fo}_2 = 0.0116$; 2) 0.0348; 3) 0.058; 4) 0.087; 5) 0.116; 6) 0.174; 7) 0.232; 8) 0.29; 9) 0.406.

We now show that the stated problem can be solved by the method of [8], according to which the nonstationary temperature field of a two-layered plate for time-varying functions $h_1(\tau)$, $h_2(\tau)$, $t_{c_1}(\tau)$, and $t_{c_2}(\tau)$ is easily obtained if the solution of the problem for constant boundary conditions, i.e., a solution of the type in [7], is known.

The determination of the roots of the characteristic equation in [7] is readily expedited with present-day computer techniques. Nevertheless, as it is only necessary to demonstrate the legitimacy of applying the method of [8] to the solution of the temperature field problem for a two-layered plate under time-varying boundary conditions, we merely use the solution of [3]. We therefore consider the temperature field problem for a two-layered plate subject to time-varying heat-transfer conditions at one boundary, considering the second boundary to be adiabatic.

The results of a calculation of the temperature variation at individual points of the two-layered plate by the method of [8] for various laws governing the variation of the boundary conditions are compared with electrosimulation (modeling) data for the same problems on a USM-1 analog computer equipped with a section for analog modeling of the time-varying values of the function $h_1(\tau)$.

We consider the problem of the nonstationary temperature field of two-layered plates composed of a highly thermal-conducting material and an insulation material with the following physical characteristics: $\lambda_1 = 36.7$ W/m · deg; $\lambda_2 = 0.227$ W/m · deg; $c_1 = 0.46 \cdot 10^3$ J/kg · deg; $c_2 = 0.734 \cdot 10^3$ J/kg · deg; $\gamma_1 = 802$ kg/m^3; $\gamma_2 = 81.5$ kg/m^3.

The plate thicknesses were chosen so that, in the notation of [3], the first two-layered plate would have parameters $\rho = 0.1$ and $\xi = 0.01$, corresponding to $\delta_1 = 0.181$ m and $\delta_2 = 0.1$ m. The second two-layered plate had parameters $\rho = 1.0$ and $\xi = 0.001$, corresponding to $\delta_1 = 0.0181$ m and $\delta_2 = 0.1$ m.

The materials and thicknesses of the layers were especially chosen to permit the values given in [3] for the roots of the characteristic equation to be used. However, since the roots of the characteristic equation were found for values of $\xi = 1/\mathrm{Bi}$ over rather large intervals, it is convenient for the analytic points to construct $\bar{\Theta} = f(\mathrm{Bi})$ from the parameter Fo_2 so as to be able subsequently to determine $\bar{\Theta} = f(\mathrm{Fo}_2)$ from the parameter Bi according to the known values of $\bar{\Theta}$ for fixed values of Fo_2 (Fig. 3).

The boundary and initial conditions for the regimes in which the analytic values calculated by the method of [8] were compared with the electrosimulation data at individual points of the two types of two-layered plates are given in Table 1. The results of a comparison of the temperature variations obtained at certain points of the two-layered plates by the approximate numerical method and electrosimulation on the USM-1 are given in Table 2.

TABLE 1. Summary of Boundary and Initial Conditions in the Analyzed Regimes

Regime	Heat-transfer intensity law, $\mathrm{Bi}_1 = f(\mathrm{Fo}_2)$	Law of temperature variation of the medium, $t_{c_1} = \varphi(\mathrm{Fo}_2)$	Coordinates of analytic point x/δ_1	Type of plate, ρ	Initial conditions t_0, °K
1	$\mathrm{Bi} = 0{,}1 \exp(3\mathrm{Fo}_2)$	$t_c = 120 - 60 \exp(-4\mathrm{Fo}_2)$	1,0	0,1	289
2	$\mathrm{Bi} = \exp(3\mathrm{Fo}_2)$	$t_c = 120 - 60 \exp(-4\mathrm{Fo}_2)$	0,667	0,1	289
3	$\mathrm{Bi} = 0{,}1 \exp(3\mathrm{Fo}_2)$	$t_c = 1000 [1 - \exp(-0{,}5\mathrm{Fo}_2)] + t_0$	1,0	0,1	298
4	$\mathrm{Bi} = \exp(3\mathrm{Fo}_2)$	$t_c = 1000 [1 - \exp(-0{,}5\mathrm{Fo}_2)] + t_0$	0,667	0,1	298
5	$\mathrm{Bi} = \exp(3\mathrm{Fo}_2)$	$t_c = t_0 [3\mathrm{Fo}_2 + \exp(3\mathrm{Fo}_2)]$	0,667	0,1	323
6	$\mathrm{Bi} = 0{,}1 \exp(3\mathrm{Fo}_2)$	$t_c = t_0 [3\mathrm{Fo}_2 + \exp(3\mathrm{Fo}_2)]$	1,0	0,1	323
7	$\mathrm{Bi} = \exp(3\mathrm{Fo}_2)$	$t_c = 473$ °K	0,667	0,1	273
8	$\mathrm{Bi} = 0{,}1 \exp(3\mathrm{Fo}_2)$	$t_c = 120 - 60 \exp(-4\mathrm{Fo}_2)$	0,33	1,0	289
9	$\mathrm{Bi} = 0{,}1 \exp(3\mathrm{Fo}_2)$	$t_c = 450$ °K	0,667	1,0	350

TABLE 2. Comparison of Temperature Variations at Points of a Two-Layered Plate According to the Approximate Numerical Method and Analog Computer Modeling in the Regimes Indicated in Table 1

Fo_2	Regime 1		2		3		4		5		6		7		8		9	
	Coordinates of point x/δ_1																	
	1,0		0,667		1,0		0,667		0,667		1,0		0,667		0,333		0,667	
	t_a, °C	t_M, °C	t_a, °C	t_M, °C	t_a, °C	t_M, °C	t_a, °C	t_M, °C	t_a, °C	t_M, °C	t_a, °C	t_M, °C	t_a, °C	t_M, °C	t_a, °C	t_M, °C	t_a, °C	t_M, °C
0,0348	18,8	19,0	30,0	29,1	25,7	25,6	27,7	27,0	50,6	50,2	50,2	50,3	62,0	62,2	37,4	36,8	124,0	124,6
0,0572	20,6	20,5	36,6	37,0	26,5	26,5	31,9	31,9	52,7	52,2	50,6	50,4	87,0	86,7	47,1	48,3	145,2	145,4
0,0875	22,6	22,3	44,3	45,0	28,05	28,0	38,9	39,4	54,4	54,4	51,0	50,6	109,5	110,0	57,1	58,5	154,7	155,7
0,116	25,1	24,9	53,2	52,4	30,2	31,0	47,7	48,4	57,8	57,6	51,7	51,4	131,0	130,2	67,0	66,7	162,7	164,0
0,175	30,7	31,0	68,5	67,1	37,3	38,5	71,0	72,5	66,8	66,8	54,3	54,2	162,0	162,0	79,5	79,8	172,0	173,0
0,231	36,9	37,3	79,5	78,1	45,8	45,1	96,0	99,0	79,2	79,6	58,4	58,6	178,0	179,0	89,2	88,3	174,5	176,0
0,291	45,0	44,9	89,0	88,1	57,0	55,6	125,0	127,0	92,7	94,2	64,8	65,4	189,0	190,0	96,9	96,2	176,5	176,9
0,408	61,7	61,2	102,5	102,3	85,5	82,0	177,0	182,0	124,1	128,6	85,0	85,6	194,0	198,0	107,2	107,0	176,7	177,0
0,582	87,2	86,0	111,0	112,8	150,0	145,0	255,0	261,0	185,0	189,0	137,0	140,6	198,0	200,0	113,0	114,8	177,0	177,0

TABLE 3. Comparison of the Temperature Variation at the Point $x/\delta_1 = 1.0$ of a Two-Layered Plate in Regime 1 According to the Approximate Numerical Method and Analog Computer Modeling

τ, h	0,31	0,52	0,78	0,99	1,04	1,3	1,56	1,98	2,08	2,60	2,97	3,12	3,64	3,95	4,16	4,68	4,94	5,20
$\Delta\tau$, h	0,31	0,21	0,26	0,21	0,05	0,26	0,26	0,42	0,1	0,52	0,37	0,15	0,52	0,31	0,21	0,52	0,26	0,26
$(t_{c_i} - t_0)$, °C	46	51	56	64	71	71	71	79	79	84	89	92	92	92	92	96	96	97
$\frac{t_{c_i} - t_0}{t_{c_{i+1}} - t_0}$	—	0,902	0,912	0,876	0,902	1,0	1,0	0,9	1,0	0,942	0,945	0,967	1,0	1,0	1,0	0,96	1,0	0,99
Bi_m	0,1	0,11	0,12	0,13	0,15	0,15	0,17	0,18	0,19	0,215	0,25	0,27	0,31	0,36	0,39	0,44	0,48	0,53
$\overline{\Theta}_{i_0}$	0	0,054	0,082	0,103	0,117	0,127	0,162	0,187	0,255	0,249	0,320	0,369	0,395	0,486	0,540	0,548	0,630	0,670
$\overline{\Theta}_{i_e}$	0,06	0,09	0,113	0,130	0,127	0,162	0,208	0,255	0,265	0,339	0,382	0,395	0,486	0,540	0,572	0,630	0,675	0,715
t_a, °C	18,8	20,6	22,6	24,3	25,1	27,5	30,7	36,1	36,9	44,5	50,0	52,4	60,6	65,6	68,7	76,5	80,7	85,2
U_M	0,03	0,045	0,063		0,089		0,150		0,213	0,289			0,452					0,700
t_M, °C	19,0	20,5	22,3		24,9		31,0		37,3	44,9			61,2					86,0

We recall that, as shown in [8], the temperature variation at any point of a body under variable boundary conditions can be obtained as the sum of the solutions (subscript f = "fixed")

$$\bar{\Theta}(\mathrm{Bi},\ \mathrm{Fo}_2)\Big|_0^{\mathrm{Fo}_2(n\Delta\tau)} = \underset{0<\mathrm{Fo}_2<\mathrm{Fo}_2(\Delta\tau)}{\bar{\Theta}(\mathrm{Bi}_1,\ \mathrm{Fo}_2)} + \underset{\mathrm{Fo}_2(\tau_{1f})<\mathrm{Fo}_2<\mathrm{Fo}_2(\tau_{1f}+\Delta\tau)}{\bar{\Theta}(\mathrm{Bi}_2,\ \mathrm{Fo}_2)}$$
$$+ \underset{\mathrm{Fo}_2(\tau_{2f})<\mathrm{Fo}_2<\mathrm{Fo}_2(\tau_{2f}+\Delta\tau)}{\bar{\Theta}(\mathrm{Bi}_3,\ \mathrm{Fo}_2)} + \cdots + \underset{\mathrm{Fo}_2\left(\tau_{(n-1)f}\right)<\mathrm{Fo}_2<\mathrm{Fo}_2\left(\tau_{(n-1)f}+\Delta\tau\right)}{\bar{\Theta}(\mathrm{Bi}_n,\ \mathrm{Fo}_2)}, \qquad (3)$$

if the functions $\mathrm{Bi} = f(\tau)$ and $t_c = \varphi(\tau)$ are replaced by piecewise step functions and the lower limits of Fo_2 in each interval of constant Bi_m are determined from the conditions guaranteeing continuous differentiability of the solution.

A detailed sample calculation of the temperature variation at the point $x/\delta_1 = 1.0$ in regime 1 of Table 1 is presented in Table 3.

The replacement of the continuous functions $\mathrm{Bi} = f(\tau)$ and $[t_c(\tau) - t_0]$ by piecewise step functions for unequal intervals of constant Bi_m and $(t_c - t_0)$ makes it possible in certain zones to obtain a more complete picture of how the variations of Bi and t_c are taken into account in the solution of the problem.

From the quantity $\bar{\Theta}_{i_e}$ at the end of each conditional interval of constant Bi_m and $(t_{c_i} - t_0)$ we determine the value of the temperature at the analytic point at given times (see Table 2) as

$$t_a = \bar{\Theta}_{i_e}(t_{c_i} - t_0) + t_0 .$$

Simultaneously we find the temperature values obtained at the analytic points by modeling of the problem on an analog computer according to the relation

$$t_M = U_M(t_c^{\max} - t_0) + t_0 .$$

An analysis of the results given in Tables 2 and 3 permits us to state that the method described in [8] for the approximate numerical solution of the problem of the nonstationary temperature field under time-varying boundary conditions is also applicable to the calculation of the temperature field of a system of bodies when solutions have been found for the problem under constant boundary conditions.

NOTATION

$a_1 = \lambda_1/c_1\gamma_1$,
$a_2 = \lambda_2/c_2\gamma_2$ are the thermal diffusivities of the high- and low-thermal-conductivity layers;
$h_1(\tau) = \alpha_1(\tau)/\lambda_1$,
$h_2(\tau) = \alpha_2(\tau)/\lambda_2$ are the heat-transfer coefficients at the boundaries of the high- and low-conductivity layers;
$t_{c_1}(\tau)$, $t_{c_2}(\tau)$ are the temperature of medium at the side of highly heat-conducting and less heat-conducting layers;
$\rho = c_2\gamma_2\delta_2/c_1\gamma_1\delta_1$;
$\xi = \lambda_2\delta_1/\lambda_1\delta_2$;
$\Delta\tau$ is the interval of constancy of Bi_m and t_{c_1};
$\mathrm{Fo}_2 = a_2\tau/\delta_2^2$ is the Fourier criterion;
$\mathrm{Bi} = h_1\delta_1$ is the Biot criterion;
U_M is the potential measured at an analytic point of the model at certain times;
$t_c^{\max}$ is the maximum temperature of the medium in the investigated problem;
t_0 is the initial temperature of the body.

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